Riemann Integration - II

Integrable Function: Let $F \in R[a,b]$ then the Function ϕ defined on $[a,b]$ as $\phi(t) = \int_a^b f(x)dx$ is called integrable function of t.

NOTE:

i) F is continuous at $x=a \Rightarrow \underset{x\to a}{Lt} f(x) = f(a)$

$$\Rightarrow \underset{x\to a}{Lt} f(a+h) = f(a)$$

$\forall \epsilon>0 \ \exists \ \delta>0$ such that $f(a+h) = f(a)| < \epsilon$, Where $|h|<\delta$

ii) F is differentiable at $a \Rightarrow \underset{x\to a}{Lt} \dfrac{f(x)-f(a)}{x-a} = f(a)$

$$\Rightarrow \underset{h\to 0}{Lt} \frac{f(a+h)-f(a)}{a+h-a} = f(a)$$

$\forall \epsilon>0 \ \exists \ \delta>0$ such that $\left|\dfrac{f(a+h)-f(a)}{h}\right| < \epsilon$

Where $|h|<\delta$.

Theorem 1: If $F\in R[a,b]$ then $\phi(t) = \int_a^t f(x)dx \ \forall \ t\in[a,b]$ is continuous on $[a,b]$

Proof: suppose $F\in R[a,b] \Rightarrow F$ is a continuous on $[a,b]$

$|f(x)| \leq k$, Where k is a real number

Given $\phi(t) = \int_a^t f(t)dx, \ \forall \ t\in[a,b]$

choose $\delta>0$ such that $\delta = \dfrac{\epsilon}{k}$

suppose $|h|<\delta$.

Now, We P.T ϕ is continuous on $[a,b]$

It is enough to P.T ϕ is continuous at 'c'.

By definition

$$|\phi(c+h) - \phi(c)| = \left| \int_a^{c+h} f(x)dx - \int_a^c f(x)dx \right|$$

$$= \left| \int_a^c f(x)dx + \int_c^{c+h} f(x)dx - \int_a^c f(x)dx \right|$$

$$= \left| \int_c^{c+h} f(x)\,dx \right|$$

$$\leq \int_c^{c+h} |f(x)|\,dx$$

$$\leq \int_c^{c+h} K\,dx$$

$$\leq \int_c^{c+h} dx$$

$$\leq K\,[x]_c^{c+h}$$

$$\leq K(c+h-c)$$

$$\leq Kh$$

$$< K\delta$$

$$< K\frac{\epsilon}{K}$$

$$< \epsilon$$

$\therefore$ ϕ is continuous at 'c'

ϕ is continuous on $[a,b]$

Theorem-2: If $F \in R[a,b]$ and. F is continuous at 'c' $c \in [a,b]$ then $\phi(t) = \int_a^t f(x)\,dx$ is derivable 'c' and $\phi(c) = f(c)$.

Proof: suppose $F \in R\ [a,b]$ and 'F' is continuous at 'c' and $\phi(t) = \int_a^t f(x)\,dx$

now, we P.T $\phi(t)$ is derivable at 'c' and since 'f' is continuous at 'c' $\forall\, \epsilon > 0\ \exists\, \delta > 0$ such that

$$|f(x) - f(c)| < \epsilon$$

where $|x-c| < \delta$

choose $|h| < \delta \longrightarrow (1)$

$$\left| \frac{\phi(c+h) - \phi(c)}{h} - f(c) \right|$$

$$\left| \frac{\int_a^{c+h} f(x)\,dx - \int_a^{c} f(x)\,dx}{h} - f(c) \right|$$

$$= \left| \frac{\int_a^{c+h} f(x)dx - \int_a^c f(x)dx}{h} - \frac{1}{h}\int_c^{c+h} f(c)dx \right|$$

$$= \frac{1}{h}\left| \int_a^c f(x)dx + \int_c^{c+h} f(x)dx - \int_c^{c+h} f(c)dx \right|$$

$$= \frac{1}{h}\left| \int_c^{c+h} [f(x) - f(c)]\,dx \right|$$

$$< \frac{1}{h}\left| \int_c^{c+h} \epsilon\, dx \right|$$

$$< \frac{1}{h}\,\epsilon \int_c^{c+h} dx$$

$$< \frac{\epsilon}{h}[x]_c^{c+h}$$

$$< \frac{\epsilon}{h}(c+h-c)$$

$$< \frac{\epsilon}{\cancel{h}}\cancel{h}$$

$$< \epsilon$$

$\therefore \phi$ is derivable at c.

Primitive of 'F':

If $f \in R[a,b]$ and there exists $\phi: [a,b] \to R$ such that $\phi'(x) = f(x)$ $\forall\, x \in [a,b]$ then ϕ is called primitive of 'f'

Fundamental theorem of integral calculates:

Statement: If $f \in R[a,b]$ and ϕ is primitive of 'f' then $\int_a^b f(x)dx = \phi(b) - \phi(a)$.

Proof: suppose $f \in R[a,b]$ and ϕ is primitive of 'f' $\Leftrightarrow \phi'(x) = f(x) \to (1)$

let $P = \{a = x_0, x_1, x_2 \dots x_n = b\}$ be a partition on $[a,b]$

Let, $I_r = [x_{r-1}, x_r]$ and $\xi_r \in I_r \Rightarrow x_{r-1} \le \xi_r \le x_r$

By definition, $\int_a^b f(x)\,dx = \underset{\|P\| \to 0}{Lt} \sum_{r=1}^{n} f(\xi_r)\,\delta_r \longrightarrow ②$

ϕ is continuous on $[a,b]$

ϕ is derivable on (a,b)

ϕ is continuous on $[x_{r-1}, x_r]$

ϕ is derivable on (x_{r-1}, x_r)

By lagranger's theorem

$\exists\ \xi_r \in (x_{r-1}, x_r)$ such that $- \phi'(\xi_r) = \dfrac{\phi(x_r) - \phi(x_{r-1})}{x_r - x_{r-1}}$

$\phi'(\xi_r)(x_r - x_{r-1}) = \phi(x_r) - \phi(x_{r-1})$

From ① $\Rightarrow f(\xi_r)\,\delta_r = \phi(x_r) - \phi(x_{r-1})$

$\Rightarrow \sum_{r=1}^{n} f(\xi_r)\,\delta_r = \sum_{r=1}^{n} \phi(x_r) - \phi(x_{r-1})$

$\Rightarrow \phi(x_1) - \phi(x_0) + \phi(x_2) - \phi(x_1) + \dots + \phi\ x_n - \phi(x_{n-1})$

$\Rightarrow -\phi(x_0) + \phi(x_n)$

$\Rightarrow \sum_{r=1}^{n} f(\xi_r)\,\delta_r = \phi(b) - \phi(a)$

$\underset{\|P\|}{Lt} \sum_{r=1}^{n} f(\xi_r)\,\delta_r = \underset{\|P\|}{Lt} [\phi(b) - \phi(a)]$

$\int_a^b f(x)\,dx = \phi(b) - \phi(a)$ [$\because$ From ②]

Using Fundemental theorem of integral calculation:

1) show that $\int_0^1 x^4\,dx = 1/5$

Sol: Let $f(x) = x^4, \forall x \in [0,1]$

$\therefore f$ is bounded on $[0,1]$

Every continuous Function is Riemann Integrable

$\Rightarrow f \in R[a,b]$

now $\phi(x) = \int x^4\,dx \Rightarrow x^5/5 \to ①$

If ϕ is primitive of 'f' then

$$\phi'(x) = f(x)$$
$$= \frac{5x}{5}$$
$$= f(x)$$

$\therefore$ Fundemental Theorem is integral

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a)$$

$$\int_0^1 x^4\,dx = \phi(1) - \phi(0)$$
$$= \frac{(1)^5}{5} - \frac{0}{5} \quad [\because \text{From } (1)]$$

$$\therefore \int_0^1 x^4\,dx = 1/5$$

2) show that $\int_0^1 \frac{1}{1+x^2}\,dx = \pi/4$

Sol: Let $f(x) = \frac{1}{1+x^2}, \forall x \in [0,1]$

$\therefore$ f is bounded on $[0,1]$

f is continuous on $[0,1]$

Every continuous function is Rieman integrable $\Rightarrow f \in R[a,b]$

now, $\phi(x) = \int \frac{1}{1+x^2}\,dx$

$$= \tan^{-1} x$$

If ϕ is primitive of 'f' then $\phi'(x) = f(x)$

$$\phi'(x) = \frac{1}{1+x^2}$$

$\therefore$ Fundamental theorem integral calculates is

$$\int_0^1 \frac{1}{1+x^2}\,dx = \phi'(1) - \phi(0)$$

$\Rightarrow \tan^{-1}(1) - \tan^{-1}(0)$

$\Rightarrow \tan^{-1}(\tan \pi/4) = \pi/4$

$\therefore \int_0^1 \frac{1}{1+x^2}dx = \pi/4$

3) Show that $\int_0^{1/2} \frac{1}{\sqrt{1-x^2}} dx = \pi/6$

sol: let $f(x) = \frac{1}{\sqrt{1-x^2}}$

f is bounded on $[0, \frac{1}{2}]$ and continuous on $[0,\frac{1}{2}]$ every continuous Function is integrable.

$\Rightarrow f \in R(0, \frac{1}{2})$

$\phi(x) = \int \frac{1}{\sqrt{1-x^2}} dx$

$\phi(x) = \sin^{-1}(x)$

If ϕ is primitive of 'f' then $\phi(x) = f(x)$

$\phi'(x) = \frac{1}{\sqrt{1-x^2}}$

$\therefore$ Fundamental theorem of integral calculus is

$$\int_0^{1/2} \frac{1}{\sqrt{1-x^2}} dx = \phi\left(\frac{1}{2}\right) - \phi(0)$$

$$= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)$$

$$= \sin^{-1}(\sin \pi/2) - \sin^{-1}(\sin(0))$$

$$= \pi/6$$

$$\therefore \int_0^{1/2} \frac{1}{\sqrt{1-x^2}} dx = \frac{\pi}{6}$$

First mean value theorem:

Statement: If $f, g \in R$ and f, g keeps same sign on $[a,b]$ then there exists $\mu \in R$ lying between inf f, sup f such that $\int_a^b f(x)g(x)\,dx$.

$= \mu \int_a^b g(x)dx.$

Proof: suppose $f, g \in R[a,b]$

case(i): let $g(x) \geq 0$

$\therefore f \in R[a,b]$

$\Rightarrow f$ is bounded on $[a,b]$

Let M, m be sup & inf of 'f' on $[a,b]$

$\Rightarrow m \leq f(x) \leq M, \forall x \in [a,b]$

$\Rightarrow m g(x) \leq f(x) g(x) \leq M g(x)$

$\Rightarrow m \int_a^b g(x)\,dx \leq \int_a^b f(x).g(x)\,dx \leq M \int_a^b g(x)\,dx$

dividing on $\int_a^b g(x)\,dx$

$$\Rightarrow m \leq \frac{\int_a^b f(x)\,g(x)\,dx}{\int_a^b g(x)\,dx} \leq M$$

Let $\int_a^b \frac{f(x)\,g(x)\,dx}{\int_a^b g(x)\,dx} = \mu;\ \forall \mu \in [M, m]$

$\int_a^b f(x)\,g(x)\,dx = \mu \int_a^b g(x)\,dx$

similarly we can prove theorem if $g(x) \leq 0, \forall x \in [a,b]$

1) Using 1st mean value theorem prove that $\frac{1}{\pi} \leq \int_0^1 \frac{\sin \pi x}{1+x^2}\,dx \leq \frac{2}{\pi}$

Sol: Let $f(x) = \frac{1}{1+x^2}$, $g(x) = \sin \pi x, \forall x \in [0,1]$

f, g are continuous function $[0,1]$

$f : g \in R[0,1]$

$g(x) = \sin \pi x \geq 0, \forall x \in (0,1)$

By F.M.V theorem $\int_a^b f(x)\, g(x)\, dx = \mu \int_a^b g(x)\, dx$

$$\int_0^1 \frac{\sin \pi x}{1+x^2}\, dx = \mu \int_0^1 \sin \pi x\, dx$$

$$= \mu \left[-\frac{\cos(\pi x)}{\pi} \right]_0^1$$

$$= \frac{\mu}{\pi} \left[-\cos \pi + \cos \pi(0) \right]$$

$$= \frac{\mu}{\pi}(1+1) = \frac{2\mu}{\pi}$$

$$\mu = \frac{\pi}{2} \int_0^1 \frac{\sin \pi x}{1+x^2}\, dx$$

f is decreasing function on $[0,1]$

$$f(x) = \frac{1}{1+x^2}$$

$\Rightarrow f(0) = \frac{1}{1+0} = 1, \quad f(1) = \frac{1}{1+1} = \frac{1}{2}$

Now let $M = 1$ and $m = \frac{1}{2}$

$$m \leq \mu \leq M$$

$$\Rightarrow \frac{1}{2} \leq \frac{\pi}{2} \int_0^1 \frac{\sin \pi x}{1+x^2}\, dx \leq 1$$

Multiplying by $\frac{2}{\pi}$

$$\Rightarrow \frac{1}{2} \cdot \frac{2}{\pi} \leq \int_0^1 \frac{\sin \pi x}{1+x^2}\, dx \leq \frac{2}{\pi}$$

$$= \frac{1}{\pi} \leq \int_0^1 \frac{\sin \pi x}{1+x^2}\, dx \leq \frac{2}{\pi}$$

Mean value theorem:

'f' is continuous on $[a,b]$ then there exist $c \in (a,b)$ such that $\int_a^b f(x)\, dx = f(c)\,(b-a)$

Proof : f is continous on $[a,b] \rightarrow$ f is bounded on $[a,b]$ and $f \in R[a,b]$ let m, M be the inf & sup of f on $[a,b]$

W.K.T $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$ f is continuous on $[a,b]$ $\Rightarrow$ and $\mu \in (m,M)$ then there exist $c \in (a,b)$ such that

$$f(c) = \mu$$

$$\int_a^b f(x)\,dx = \mu(b-a)$$

$$= f(c)(b-a)$$

Problems :

1) show that $\frac{1}{4} < \int_0^{1/4} \frac{dx}{\sqrt{1-x^2}} < \frac{1}{\sqrt{15}}$, using mean value theorem.

Sol: Let $f(x) = \frac{1}{\sqrt{1-x^2}}$ defined on $[0, \frac{1}{4}]$

W.K.T $f(x)$ is continuous on R and hence in particular it is continous on $[0, \frac{1}{4})$

$\therefore f(x)$ is integrable on $[0, \frac{1}{4})$

$\therefore$ By mean value theorem there exist $c \in (0, \frac{1}{4})$ such that $\int_0^{1/4} \frac{1}{\sqrt{1-x^2}}\,dx = f(c)\left(\frac{1}{4} - 0\right)$

$$= \frac{f(c)}{4}$$

$$= \frac{1}{4}\left(\frac{1}{\sqrt{1-c^2}}\right) \rightarrow ①$$

$c \in (0, \frac{1}{4}) \rightarrow 0 < c < \frac{1}{4}$

$\Rightarrow 0 < c^2 < \frac{1}{16}$

$\Rightarrow 0 > -c^2 > -\frac{1}{16}$

$\Rightarrow 1 > 1-c^2 > 1-\frac{1}{16}$

$\Rightarrow 1 > 1-c^2 > \frac{15}{16}$

$\Rightarrow 1 > \sqrt{1-c^2} > \frac{\sqrt{15}}{4}$ $\qquad -\frac{1}{4} < \int_0^{1/4} \frac{dx}{\sqrt{1-x^2}} < \frac{1}{\sqrt{15}}$

$\Rightarrow 1 < \frac{1}{\sqrt{1-c^2}} < \frac{4}{\sqrt{15}}$

$\Rightarrow \frac{1}{4} < \frac{1}{4}\frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{15}}$ [∵ From ①]

2) Using M.V.T for $\int_0^{\pi/4} \sec x\,dx$ prove that $\frac{\pi}{4} \leq \int_0^{\pi/4} \sec x\,dx \leq \frac{\pi}{2\sqrt{2}}$

Sol: Let $f(x) = \sec x$ defined on $[0, \frac{\pi}{4})$

W.K.T $f(x)$ is continous on $[0, \frac{\pi}{4}]$

$f(x)$ is bounded and integrable on $[0, \pi/4]$

By M.V.T there exist $c \in [0, \frac{\pi}{4}]$.

Such that $\int_0^{\pi/4} \sec x\,dx = f(c)\left(\frac{\pi}{4} - 0\right)$

$= \sec c\,(\pi/4)$

$= \frac{\pi}{4}\sec c \rightarrow ①$

$c \in (0, \frac{\pi}{4}) \Rightarrow 0 \leq c \leq \frac{\pi}{4}$

$\Rightarrow \cos 0 \leq \cos c \leq \cos\frac{\pi}{4}$

$1 \geq \cos c \geq \cos\frac{\pi}{4}$

$1 \leq \frac{1}{\cos c} \leq \sqrt{2}$

$1 \leq \sec c \leq \sqrt{2}$

$\frac{\pi}{4} \leq \frac{\pi}{4}\sec c \leq \frac{\pi}{4}\sqrt{2}$

$\frac{\pi}{4} \leq \int_0^{\pi/4} \sec x\,dx \leq \frac{\pi}{2}\sqrt{2}$ [∵ From ①]

<u>Theorem 1:</u> If $f \in R[a,b]$, $\overline{\int_a^b} f(x)dx = \int_a^b f(x)dx = -\underline{\int_a^b} f(x)dx$.

<u>Proof:</u> Let $P = \{a = x_0, x_1, \ldots x_n = b\}$ be a partition on $[a,b]$

let m_r, M_r be the inf & sup of 'f' on $I_r = [x_{r-1}, x_r]$

since f is bounded on $[a,b]$, f is also bounded on $[a,b]$

$$\therefore \text{Inf}(-f) = -\sup f = -M_r \text{ on } I_r \text{ and}$$

$$\sup(-f) = -\inf f = -m_r \text{ on } I_r.$$

$$\therefore L(P,f) = \sum_{r=1}^{n} (-M_r)\,\delta_r$$

$$= -\sum_{r=1}^{n} M_r \delta_r = -U(P,f)$$

$$U(P,f) = \sum_{r=1}^{n} (-m_r)\,\delta_r$$

$$= -\sum_{r=1}^{n} m_r \delta_r = -L(P,f)$$

$$\therefore \overline{\int_a^b} (-f)(x)\,dx = \inf\{U(P,-f) \,/\, P \in \wp[a,b]\}$$

$$= \inf\{-L(P,f) \,/\, P \in \wp[a,b]\}$$

$$= -\sup\{L(P,f) \,/\, P \in \wp[a,b]\}$$

$$= -\int_a^b f(x)\,dx = -\underline{\int_a^b} f(x)\,dx$$

$$\underline{\int_a^b} f(x)dx = \sup\{L(P,-f) \,/\, P \in \wp[a,b]\}$$

$$= \sup\{-U(P,f) \,/\, P \in \wp[a,b]\}$$

$$= -\inf\{U(P,f) \,/\, P \in \wp[a,b]\}$$

$$= -\overline{\int_a^b} f(x)dx = -\int_a^b f(x)dx$$

$\therefore \overline{\int_a^b} (-f)(x)\,dx = \int_a^b \underline{\int_a^b} (-f)(x)\,dx = -\int_a^b f(x)\,dx.$

Theorem-2: If $f \in R[a,b]$ and $K \in R$, then $Kf \in R[a,b]$ and $\int_a^b (Kf)x\,dx = K\int_a^b f(x)\,dx$.

Proof: Since $f \in R[a,b]$, $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$

case (i): let $K \geq 0$

Let $P = \{a = x_0, x_1, x_2, \ldots\ldots x_n = b\}$ be a partition of $[a,b]$

Let $\inf f = m_r$, $\sup f = M_r$ in $I_r = [x_{r-1}, x_r]$

f is bounded on $[a,b] \Rightarrow Kf$ is bounded on $[a,b]$

$\therefore \operatorname{Inf}(K,f) = K \inf f = K m_r$

$\sup(K,f) = K \sup f = K M_r$

$U(P,Kf) = \sum_{r=1}^{n} (K M_r)\,\delta_r = K\,U(P,f)$ and

$L(P,Kf) = \sum_{r=1}^{n} (K m_r)\,\delta_r = K\,L(P,f)$

$\therefore \overline{\int_a^b} K f(x)\,dx = \inf\{U(P,Kf) \mid P \in \wp[a,b]\}$

$= K \inf\{U(P,f) \mid P \in \wp[a,b]\}$

$= K \overline{\int_a^b} f(x)\,dx = K\int_a^b f(x)\,dx$

Also, $\underline{\int_a^b} K f(x)\,dx = \sup\{L(P,Kf) \mid P \in \wp[a,b]\}$

$= K \sup\{L(P,f) \mid P \in \wp[a,b]\}$

$= K \cdot \underline{\int_a^b} f(x)\,dx = K\int_a^b f(x)\,dx$

$\therefore \overline{\int_a^b} (Kf)(x)\,dx = \underline{\int_a^b} (Kf)(x)\,dx = K\int_a^b f(x)\,dx$

$\therefore Kf \in R[a,b]$ and $\int_a^b (Kf)(x)\,dx = K\int_a^b f(x)\,dx$.

Case(ii): Let $K < 0$

put $K = -L$ where $L > 0$, then Kf is equal to $L(-f)$.

$f \in R[a,b] \Rightarrow -f \in R[a,b]$

By case(i) $L > 0$; $-f \in R[a,b]$

$\Rightarrow L(-f) \in R[a,b]$

$\Rightarrow Kf \in R[a,b]$

Also $\int_a^b (Kf)(x)\,dx = \int_a^b L(-f)(x)\,dx$

$= L \int_a^b f(x)\,dx$

$= L(-1) \int_a^b f(x)\,dx$

$= -L \int_a^b f(x)\,dx$

$= K \int_a^b f(x)\,dx$

Theorem-3: If $f \in R[a,b]$ then $|f| \in R[a,b]$

Proof: $f \in R[a,b]$

$\Rightarrow$ for given $\epsilon > 0$ there exists a partition

$P = \{a = x_0, x_1, \dots x_n = b\}$ such that

$0 \le U(P,f) - L(P,f) < \epsilon \rightarrow (1)$

f is bounded on $[a,b] \Rightarrow |f(x)| < K$, $K \in R$, $\forall x \in R\ [a,b]$

$\Rightarrow |f|$ is bounded on $[a,b]$

Let m_r, M_r be the inf and sup of f on I_r and m_r', M_r' be the inf and sup of $|f|$ on I_r.

For each $\alpha, \beta \in I_r$ $\big| |f|(\alpha) - |f|(\beta) \big| = \big| |f(\alpha)| - |f(\beta)| \big| \le |f(\alpha) - f(\beta)|$

$\therefore M_r' - m_r' \le M_r - m_r$ for $r = 1, 2 \dots n$

$$U(P,|f|) - L(P,|f|) = \sum_{r=1}^{n} (M_r' - m_r')\,\delta_r \le \sum_{r=1}^{n} (M_r - m_r)\,\delta_r$$

$$U(P,|f|) - L(P,|f|) < \epsilon$$

$$|f| \in R[a,b]$$

NOTE: The converse of the theorem is not true

i.e, $|f|$ is integrable on $[a,b]$, f need not be integrable on $[a,b]$

consider $f : [a,b] \to R$ defined as $f(x) = 1,\ x \in Q$; $f(x) = -1$; $x \in R - Q$

Let $P = \{a = x_0, x_1, \dots x_n = b\}$ be a partition of $[a,b]$

$$\overline{\int_a^b} f(x)\,dx = \inf\{U(P,f) / P \in \wp[a,b]\} = \inf\left(\sum_{r=1}^{n} \delta_r\right)$$

$$= \inf(b-a) = b-a$$

$$\underline{\int_a^b} f(x)\,dx = \sup\{L(P,f) / P \in \wp[a,b]\}$$

$$= \sup\left(\sum_{r=1}^{n} (-1)\,\delta_r\right)$$

$$= \sup\{-(b-a)\}$$

$$\cancel{= -(b-a)}$$

$$= -(b-a)$$

$$\overline{\int_a^b} f(x)\,dx \ne \underline{\int_a^b} f(x)\,dx$$

$$\therefore f \notin R[a,b]$$

But $|f(x)| = |f|(x) = 1\ \forall x \in R$

Since $|f|$ is constant Function, $|f| \in R[a,b]$

* If $f \in R[a,b]$ then $|f| \in R[a,b]$. Is converse true or not.

Theorem-4: If $f, g \in R[a,b]$ then $f+g \in R[a,b]$ & $\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

Proof : f, g are bounded on [a,b]

$\Rightarrow$ f+g is bounded on [a,b]

let $\epsilon > 0$, $f \in R[a,b] \Rightarrow$ there exists $\delta_1 > 0$ such that

$U(P_1, f) - L(P_1, f) < \frac{\epsilon}{2}$ with $\|P_1\| < \delta_1 \longrightarrow ①$

$f \in R[a,b] \Rightarrow$ there exists $\delta_2 > 0$ such that

$U(P_2, g) - L(P_2, g) < \frac{\epsilon}{2}$ with $\|P_2\| < \delta_2 \longrightarrow ②$

let $P = P_1 \cup P_2$

then $\|P\| < \|P_1\|$ or $\|P_2\|$ and hence

$\|P_1\| < \delta_1$, $\|P\| < \delta_2$

The condition (1) & (2) are true for partition P.

We have, $W(P, f+g) = U(P, f+g) - L(P, f+g)$

$\leq \{U(P,f) - L(P,f)\} + \{U(P,g) - L(P,g)\}$

$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

$0 \leq W(P, f+g) < \epsilon$ with $\|P\| < \delta$

$\therefore f+g \in R[a,b]$

$f \in R[a,b] \Rightarrow \int_a^b f(x)dx = \lim_{\|P\| \to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r$ and

$g \in R[a,b] \Rightarrow \int_a^b g(x)dx = \lim_{\|P\| \to 0} \sum_{r=1}^{n} g(\xi_r)\delta_r$

But $\lim_{\|P\| \to 0} \sum_{r=1}^{n} (f+g)(\xi_r)(\delta_r) = \lim_{\|P\| \to 0} \sum_{r=1}^{n} \{f(\xi_r) + g(\xi_r)\}\delta_r$

$$= \lim_{\|P\| \to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r + \lim_{\|P\| \to 0} \sum_{r=1}^{n} g(\xi_r)\delta_r$$

$$\therefore \int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

<u>NOTE</u>: If $f, g \in R[a,b]$ then ~~f,g e~~ $f-g \in R[a,b]$ & $\int_a^b (f-g)x\,dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

<u>Theorem-5</u>: If $f \in R[a,b]$ then $f^2 \in R[a,b]$

<u>Proof</u>: By theorem ③ $f \in R[a,b] \Rightarrow |f| \in R[a,b]$.

f is bounded on $[a,b] \Rightarrow |f|$ is bounded on $[a,b]$

$\Rightarrow |f|^2 = f^2$ is bounded on $[a,b]$

since $f^2 = |f|^2$, without loss of generality we can suppose $f \geq 0$

Let sup f in $[a,b] = M > 0$

Let $\epsilon > 0$

$f \in R[a,b]$

$\Rightarrow \exists\, P \in \wp[a,b]$ such that $\sum_{r=1}^{n} (M_r - m_r)\delta_r = U(P,f) - L(P,f) < \frac{\epsilon}{2M+1}$

Let $\inf(f^2) = m_r^2$ & $\sup(f^2) = M_r^2$ In I_r

$$\begin{aligned}
\therefore U(P,f^2) - L(P,f^2) &= \sum_{r=1}^{n} (M_r^2 - m_r^2)\delta_r \\
&= \sum_{r=1}^{n} (M_r - m_r)(M_r + m_r)\delta_r \\
&\leq \sum_{r=1}^{n} (M_r - m_r)(M + M)\delta_r \\
&\leq 2M \sum_{r=1}^{n} (M_r - m_r)\delta_r \\
&< 2M \frac{\epsilon}{2M+1} < \epsilon
\end{aligned}$$

$$U(P,f^2) - L(P,f^2) < \epsilon$$